

EXTREMAL  $\infty$ -QUASICONFORMAL IMMERSIONS

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ABSTRACT. Given a manifold  $\mathcal{M}_0 \subseteq \mathbb{R}^N$ , we consider the problem of finding a Riemannian manifold  $(\mathcal{M}, g)$  with minimal dilation such that  $\partial\mathcal{M} = \partial\mathcal{M}_0$ . The dilation functional is the  $L^\infty$  norm of the trace of the Distortion Tensor  $\mathbf{G} := \det(g)^{-1/n} g$ . This extends the classical Teichmüller Problem. Motivated by our work on vector-valued Calculus of Variations in  $L^\infty$  [K1, K2, K3], we work on this problem by advancing the  $L^\infty$  theory of Capogna-Raich [CR] to Extremal Quasiconformal immersions  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ , i.e. solutions to

$$(1) \quad Q_\infty u := \left( K_P \otimes K_P + [K_P]^\perp K_{PP} \right) (Du) : D^2 u = 0$$

where  $K(P) = \frac{|P|^2}{\det(P^\top P)^{1/n}}$ ,  $K_P$  denotes derivative and  $[K_P]^\perp$  is projection on its nullspace. (1) is the “Euler-Lagrange PDE” of  $\|K(Du)\|_{L^\infty(\Omega)}$  for the variational notion of  $\infty$ -Minimality, which is Rank-One Local Minimality with “Minimally Distorted Area”. Nonconvexity of  $K$  and appearance of interfaces where  $[K_P]^\perp$  gets discontinuous cause extra difficulties. For  $n = 2$  interfaces disappear and results are stronger. In particular, we disprove a conjecture of Capogna-Raich on the sufficiency of (1) for their stronger variational notion. Moreover, we show their PDE is a part of (1), the projection along  $K_P(Du)$ .

## 1. INTRODUCTION

Let  $\mathcal{M}_0$  be a topological submanifold of  $\mathbb{R}^N$  with boundary. In this paper we are interested in the problem of finding a Riemannian manifold  $(\mathcal{M}, g)$  which has *minimal dilation* and satisfies  $\partial\mathcal{M} = \partial\mathcal{M}_0$ . In this setting, dilation is a functional on  $L^\infty(\mathcal{M}, \otimes^{(2)} T^* \mathcal{M})$ , defined as the  $L^\infty$  norm of the trace of the Distortion Tensor

$$(1.1) \quad \mathbf{G} := \frac{g}{\det(g)^{1/n}}.$$

This problem is an extension of the classical *Teichmüller Problem* (see [T, AIM, AIMO]). The scaling in (1.1) is such that  $\mathbf{G}$  is invariant under conformal transformations and, as we explain later, the geometric meaning of  $\text{tr}(\mathbf{G})$  being “minimal” is that “geometry is distorted as less as possible”. As a first step, we consider a simplified problem for the case of immersions  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  with prescribed boundary values on  $\partial\Omega$ . Then, the dilation functional for  $\mathcal{M} = u(\Omega)$  becomes

$$(1.2) \quad K_\infty(u, \Omega) := \|K(Du)\|_{L^\infty(\Omega)},$$

$$(1.3) \quad K(P) := \begin{cases} \frac{|P|^2}{\det(P^\top P)^{1/n}}, & \text{on } S^+, \\ +\infty, & \text{on } \mathbb{R}^N \otimes \mathbb{R}^n \setminus S^+. \end{cases}$$

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$K$  will be called the *dilation function*. In (1.3),  $|P| = \text{tr}(P^\top P)^{1/2}$  is the Euclidean norm on  $\mathbb{R}^N \otimes \mathbb{R}^n$  and

$$(1.4) \quad S^+ := \left\{ P \in \mathbb{R}^N \otimes \mathbb{R}^n : \det(P^\top P) > 0 \right\}.$$

Important objects of Geometric Topology related to (1.2) arise for  $n = N$ . Homeomorphisms  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  in  $W_{loc}^{1,n}(\Omega)^N$  which satisfy  $K_\infty(u, \Omega) < \infty$  are called *Quasiconformal Maps* and constitute a class of maps well studied in the literature; see for example [Ah2, B, G, S, V].  $L^p$  averages of Quasiconformal maps, that is weakly differentiable homeomorphisms for which  $\|K(Du)\|_{L^p(\Omega)} < \infty$  have also been systematically considered. *Conformal maps*, namely those homeomorphisms  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  in  $C^1(\Omega)^N$  which satisfy  $Du^\top Du = \frac{1}{n}|Du|^2 I$  form a special important class of Quasiconformal maps since for those  $K(Du)$  is constant and equals  $n$ . Conformal maps preserve *angles*, but not necessarily *lengths* and hence distort the geometry of  $\Omega$  in a controlled fashion. However, by Liouville's rigidity theorem, when  $n \geq 3$  the only conformal maps that exist are compositions of rotations, dilations, and the inversion  $x \mapsto x/|x|^2$ . Hence, quasiconformal maps for which  $K(Du)$  is merely bounded relax conformality but still deform  $\Omega$  to  $u(\Omega)$  in a fairly controlled fashion.

The problem with Quasiconformal maps is that too little information on their structure is provided by a mere norm bound, and the same holds for the *finite distortion problem* when one restricts attention to minimizers of the dilation functional. The subtle point is that (1.2) is *nonlocal*, in the sense that with respect to the  $\Omega$  argument (1.2) is not a measure. Simple examples certify that minimizers over a domain with fixed boundary values are not local minimizers over subdomains and the direct method of Calculus of Variations when applied to (1.2) generally does not produce PDE solutions.

In the very recent work, Capogna and Raich [CR], remedied this problem by “optimizing” Quasiconformal maps. The idea is to consider an appropriate non-standard  $L^\infty$  variational problem for (1.2) and derive a PDE governing *Extremal Quasiconformal Maps* that can be used as platform for their qualitative study. Motivated by the classical results of Aronsson [A1, A2] on *Calculus of Variations in  $L^\infty$* , they developed a PDE theory in  $L^\infty$  for *extremal quasiconformal maps*. The essence of this approach is the following: let  $Q_p u = 0$  be the Euler-Lagrange system of functional  $\|K(Du)\|_{L^p(\Omega)}$ . Then, at least formally  $Q_p$  tends to a certain operator  $Q_\infty$  and  $\|K(Du)\|_{L^p(\Omega)}$  tends to  $\|K(Du)\|_{L^\infty(\Omega)}$ , both as  $p \rightarrow \infty$ . The operator  $Q_\infty$  defines a quasilinear 2nd order system in non-divergence form. However, it is not a priori clear that the following rectangle “commutes”

$$(1.5) \quad \begin{array}{ccc} \|K(Du)\|_{L^p(\Omega)} & \longrightarrow & Q_p u = 0 \\ \downarrow p \rightarrow \infty & & \downarrow p \rightarrow \infty \\ \|K(Du)\|_{L^\infty(\Omega)} & \dashrightarrow & Q_\infty u = 0 \end{array}$$

so that  $Q_\infty$  has a variational structure with respect to  $K_\infty$ , in the sense that appropriately defined minimizers of  $K_\infty$  in the class of orientation-preserving diffeomorphisms  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  solve  $Q_\infty u = 0$ . In such an event,  $Q_\infty u = 0$  will play the role of “Euler-Lagrange PDE” for the dilation functional. This turns out to be the case, though. Among other far-reaching contributions which include a deep study of dilations of extensions up to the boundary and quasiconformal gradient flows, Capogna and Raich introduced in [CR] a localized minimality notion

for (1.2) and proved that those local minimizers among “competitors” indeed solve the formally derived PDE.

Simultaneously and independently, the author, also inspired by Aronsson’s work and the successful modern evolution of the field of Calculus of Variations in  $L^\infty$  (see for example [C]), initiated the development of vector-valued Calculus of Variations in  $L^\infty$  for general supremal functionals in [K1, K2] with particular emphasis to the model functional  $\|Du\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |Du|$  in [K3]. For a Hamiltonian  $H \in C^2(\mathbb{R}^N \otimes \mathbb{R}^n)$  and the respective supremal functional

$$(1.6) \quad E_\infty(u, \Omega) := \|H(Du)\|_{L^\infty(\Omega)},$$

the related *Aronsson PDE system* which plays the role of “Euler-Lagrange PDE” for (1.6) is

$$(1.7) \quad A_\infty u := \left( H_P \otimes H_P + H[H_P]^\perp H_{PP} \right) (Du) : D^2 u = 0.$$

Here  $[H_P(Du(x))]^\perp$  is the projection on the nullspace of  $H_P(Du(x))^\top : \mathbb{R}^N \rightarrow \mathbb{R}^n$ , and  $H_P, H_{PP}$  denotes derivatives (for details see Preliminaries 2). The special case of  $H(P) = |P|^2$  leads to the important  $\infty$ -Laplacian

$$(1.8) \quad \Delta_\infty u := \left( Du \otimes Du + |Du|^2 [Du]^\perp \otimes I \right) : D^2 u = 0.$$

System (1.7) is a quasilinear 2nd order system in non-divergence form which arises in the limit of the Euler-Lagrange system of the  $L^p$  functional  $\|H(Du)\|_{L^p(\Omega)}$  as  $p \rightarrow \infty$ . In the scalar case of  $n = 1$  the normal coefficient of (1.8)  $|Du|^2 [Du]^\perp$  vanishes, and the same holds for submersions in general. The scalar  $\infty$ -Laplacian then becomes  $Du \otimes Du : D^2 u = 0$ .

Unlike the scalar case of  $n = 1$ , in the full vector case of (1.7) intriguing phenomena appear. Except for the emergence of “singular solutions” to (1.7), a further difficulty not present in the scalar case is that (1.7) has *discontinuous coefficients* even for  $C^\infty$  solutions. There exist  $C^\infty$  solutions whose rank of  $H_P(Du)$  is not constant: such an example on  $\mathbb{R}^2$  for (1.8) is given by  $u(x, y) = e^{ix} - e^{iy}$  which is  $\infty$ -Harmonic near the origin and has  $\text{rk}(Du) = 1$  on the diagonal, but it has  $\text{rk}(Du) = 2$  otherwise and hence the projection  $[Du]^\perp$  is discontinuous. In general,  $\infty$ -Harmonic maps present a *phase separation*, thoroughly studied for  $n = 2 \leq N$  in [K2]. On each phase the dimension of the tangent space is constant and these phases are separated by *interfaces* whereon the rank of  $Du$  “jumps” and  $[Du]^\perp$  gets discontinuous.

In this paper we work towards the problem mentioned in the beginning by extending the theory of [CR] to the case of immersions  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  and in the same time we elaborate it and make it more efficient in certain respects. First of all, we allow for positive codimension  $N - n$  and take into account the exterior geometry of immersions. Moreover, our maps are local diffeomorphisms onto their images, but in our analysis we do *not* impose the global topological constraint that our maps are homomorphisms onto their image and allow for self-intersections. However, *all* our results and notions are still valid and with the exact same proofs in this restricted class. For distinction, we introduce the following terminology: *an immersion  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  in  $C^1(\Omega)^N$  is called  $p$ -Quasiconformal when  $\|K(Du)\|_{L^p(\Omega)} < \infty$ ,  $1 \leq p \leq \infty$ .* We begin by repeating part of the program of [CR] to the extended case. After some introductory material is Section 2,

in Section 3 we calculate the PDE system which *Extremal  $p$ -Quasiconformal immersions* satisfy (equations (3.23), (3.24)), that is the Euler-Lagrange system of  $K_p(u, \Omega) := \|K(Du)\|_{L^p(\Omega)}$ . Then, in Section 4 we formally derive in the limit as  $p \rightarrow \infty$  the Aronsson PDE system which *Extremal  $\infty$ -Quasiconformal immersions*  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  satisfy, that is the Aronsson system related to (1.2):

$$(1.9) \quad Q_\infty u := \left( K_P \otimes K_P + [K_P]^\perp K_{PP} \right) (Du) : D^2 u = 0$$

where the derivatives of the dilation are given by

$$(1.10) \quad K_P(Du) = 2Du \frac{g^{-1}S(g)}{\det(g)^{1/n}}$$

$$(1.11) \quad K_{PP}(Du) = 2 \left( I \otimes \frac{g^{-1}S(g)}{\det(g)^{1/n}} + Du \otimes Du : \frac{g^{-1}E}{\det(g)^{1/n}} \right) + O(Du).$$

Here  $g = Du^\top Du$ ,  $S$  is the Ahlfors operator given by (2.7),  $E$  is a constant tensor given by (3.11) and  $O(Du)$  is a tensor annihilated by  $[K_P(Du)]^\perp$  and does not appear in the PDE system (1.9) (for details see Lemmas 3.1, 3.2). The derivation has overlaps with the respective in [K1], but is not a direct consequence since we utilize the specific structure of the Hamiltonian (1.3). By restricting ourselves to  $n = N$  and employing Lemma 4.2 to relate the seemingly different system (1.9) to that of [CR], we see that the derivation as  $p \rightarrow \infty$  in [CR] is incomplete and their PDE is only a part of (1.9). System (1.9) consists of two systems whose defining vector-valued nonlinearities are normal to each other:

$$(1.12) \quad K_P(Du) \otimes K_P(Du) : D^2 u = 0,$$

$$(1.13) \quad [K_P(Du)]^\perp K_{PP}(Du) : D^2 u = 0.$$

System (1.12) is the “tangential” part in (the range of the projection)  $[K_P(Du)]^\top$  and system (1.13) is the “normal” part in  $[K_P(Du)]^\perp$  (see Figure 1). The reason for this terminology is that  $[Du]^\top$  is (the projection on) the tangent bundle of the immersion,  $[Du]^\perp$  is its normal bundle and by (1.10) we have that  $[K_P(Du)]^\top \subseteq [Du]^\top$ .

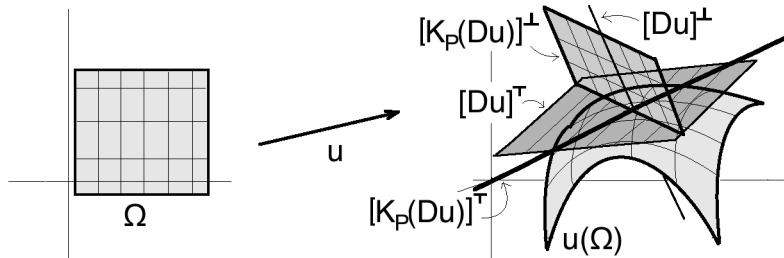


Figure 1.

The derivation in [CR] has *lost information of extremal quasiconformality along directions in  $[K_P(Du)]^\perp$  and reveals only system (1.12)*. System (1.13) appears also in zero-codimension when  $n = N$  since generally  $K_P(Du)$  does not have rank equal to  $n$ , although by assumption the rank of  $Du$  equals  $n$ . More importantly, *when the rank of  $K_P(Du)$  becomes nonconstant, the coefficients of (1.9) become discontinuous*. This leads to the appearance of interfaces whereon the projection  $[K_P(Du)]^\perp$  becomes discontinuous. These interfaces are boundaries of the different phases to which Extremal  $\infty$ -Quasiconformal maps naturally separate.

In Section 5 we move to the variational structure of Extremal  $\infty$ -Quasiconformal maps. Inspired from [K3], we introduce the variational notion of  $\infty$ -Minimal Dilation, which is Rank-One Locally Minimal Dilation with “Minimally Distorted Area” of  $u(\Omega)$  (Definition 5.1). Rank-one locally minimal dilation requires that an immersion is a local minimizers for the dilation functional when the “set of competitors” is the one obtained by taking essentially scalar local variations with fixed zero boundary values (Figure 2). Minimally distorted area means that the immersion is a local minimizer where the “set of competitors” is the one obtained by taking variations along sections of the normal vector bundle  $[K_P(Du)]^\perp$  over  $u(\Omega)$  with free boundary values (Figure 3). The appearance of interfaces where the dimension of  $[K_P(Du)]^\perp$  jumps causes substantial difficulties, even in the very definition of the minimality notion. Our first main result is Theorem 5.2, wherein we prove that  $\infty$ -Quasiconformal maps with  $\infty$ -Minimal Dilation are Extremal, *at least off the interfaces of discontinuities in the coefficients*. This result follows closely Theorem 2.1 in [K1] and Theorem 2.2 in [K3], but nonconvexity of (1.3), appearance of discontinuities in (1.9) and the necessity of restriction to specific variations create complications not present in the results just quoted. We note that the rank-one minimality notion gives rise to the tangential system and the condition on the minimality of the area gives rise to the normal system.

In Section 6 we study some geometric aspects of (1.9) and of the interfaces of its solutions. In Subsection 6.1 we show that system (1.9) has a “geometric” rather coordinate-free reformulation, at least off interfaces of discontinuities. More precisely, (1.12) and (1.13) are respectively equivalent to

$$(1.14) \quad S(\mathbf{G})D(\text{tr}(\mathbf{G})) = 0,$$

$$(1.15) \quad \mathbb{B}^\perp : (\text{tr}(\mathbf{G}))_P = 0,$$

where  $\mathbf{G}$  is given by (1.1) for  $g = Du^\top Du$  and  $\mathbb{B}^\perp$  is a “generalized 2nd fundamental form” with respect to normal sections valued in  $[K_P(Du)]^\perp$ . If  $K_P(Du)$  has full rank  $n$ , then  $[K_P(Du)]^\perp$  coincides with the normal bundle  $[Du]^\perp$  of the immersion and  $\mathbb{B}^\perp$  reduces to the standard object. System (1.14) is quite “metrically invariant” but system (1.15) depends on the exterior geometry and measures the “shape of  $u(\Omega)$ ”. In Subsection 6.2, by assuming some a priori local  $C^1$  regularity on the interfaces but with possible self-intersections, we prove an identity which shows that the covariant gradient of  $[K_P(Du)]^\perp$  along the interface is differentiable when projected along  $K_P(Du)$ .

In Section 7 we turn our attention to the converse statement of that in Theorem 5.2, that is the sufficiency of (1.9) for the variational notion of  $\infty$ -Minimal Dilation. Nonconvexity of (1.3) and the resemblance to similar phenomena in *Minimal Surfaces* leaves little hope for system (1.13) to be sufficient for minimally distorted area. However, in Proposition 7.2 we establish that *when  $n = 2 \leq N$  there is a triple equivalence among solutions of (1.12), the condition the dilation (1.3) to be constant and the immersion to have rank-one locally minimal dilation*. This result relates directly to the two-dimensional results in [Ah1, B, H]. In particular, *when  $n = 2$  interfaces disappear and the coefficients of (1.9) become continuous*.

Moreover, as a consequence of Example 7.5 which certifies that rank-one locally minimal dilation is *strictly weaker* than the variational notion utilized in [CR] with respect to general vector-valued variations (among competitors), *we disprove the conjecture of Capogna-Raich on the sufficiency of (1.3) explicitly stated in p. 855*.

Finally, at the end of Section 7 we loosely discuss the much more complicated case when  $n \geq 3$ . In this case results are less sharp. Although it is hardly conclusive, it seems that dilation may not be constant but we do believe that (1.12) is still sufficient for rank-one locally minimal dilation.

Throughout this paper, as in [CR] and also in [K1, K2, K3], we restrict our analysis to the unnatural class of  $C^2$  solutions to the Aronsson system. This is only the first step in our study and we can not go much further without an appropriate “weak” theory of nondifferentiable solutions for (1.9). In the forthcoming paper [K4] we introduce such an approach which applies to fully nonlinear PDE systems and in this setting therein we consider the problem of existence for the  $\infty$ -Laplacian (1.8). This opens up the way towards the rigorous efficient study of nonsmooth Extremal Quasiconformal maps.

## 2. PRELIMINARIES.

Throughout this paper we reserve  $n, N \in \mathbb{N}$  for the dimensions of Euclidean spaces and  $\mathbb{S}^{N-1}$  denotes the unit sphere of  $\mathbb{R}^N$ . Greek indices  $\alpha, \beta, \gamma, \dots$  run from 1 to  $N$  and Latin  $i, j, k, \dots$  form 1 to  $n$ . The summation convention will always be employed in repeated indices in a product. Vectors are always viewed as columns and we differentiate along rows. Hence, for  $a, b \in \mathbb{R}^n$ ,  $a^\top b$  is their inner product and  $ab^\top$  equals  $a \otimes b$ . If  $V$  is a vector space, then  $\mathbb{S}(V)$  denotes the symmetric linear maps  $T : V \rightarrow V$  for which  $T = T^\top$ . If  $u = u_\alpha e_\alpha : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  is in  $C^2(\Omega)^N$ , the gradient matrix  $Du$  is viewed as  $D_i u_\alpha e_\alpha \otimes e_i : \Omega \rightarrow \mathbb{R}^N \otimes \mathbb{R}^n$  and the Hessian tensor  $D^2u$  as  $D_{ij}^2 u_\alpha e_\alpha \otimes e_i \otimes e_j : \Omega \rightarrow \mathbb{R}^N \otimes \mathbb{S}(\mathbb{R}^n)$ . The Euclidean (Frobenius) norm on  $\mathbb{R}^N \otimes \mathbb{R}^n$  is  $|P| = (P_{\alpha i} P_{\alpha i})^{\frac{1}{2}} = (\text{tr}(P^\top P))^{\frac{1}{2}}$ . We also introduce the following *contraction operation* for tensors which extends the Euclidean inner product  $P : Q = \text{tr}(P^\top Q) = P_{\alpha i} Q_{\alpha i}$  of  $\mathbb{R}^N \otimes \mathbb{R}^n$ . Let “ $\otimes^{(r)}$ ” denote the  $r$ -fold tensor product. If  $S \in \otimes^{(q)} \mathbb{R}^N \otimes^{(s)} \mathbb{R}^n$ ,  $T \in \otimes^{(p)} \mathbb{R}^N \otimes^{(s)} \mathbb{R}^n$  and  $q \geq p$ , we define a tensor  $S : T$  in  $\otimes^{(q-p)} \mathbb{R}^N$  by

$$(2.1) \quad S : T := (S_{\alpha_q \dots \alpha_p \dots \alpha_1 i_s \dots i_1} T_{\alpha_p \dots \alpha_1 i_s \dots i_1}) e_{\alpha_q} \otimes \dots \otimes e_{\alpha_{p+1}}.$$

For example, for  $s = q = 2$  and  $p = 1$ , the tensor  $S : T$  of (2.1) is a vector with components  $S_{\alpha \beta i j} T_{\beta i j}$  with free index  $\alpha$  and the indices  $\beta, i, j$  are contracted. In particular, in view of (2.1), the 2nd order linear system

$$(2.2) \quad A_{\alpha i \beta j} D_{ij}^2 u_\beta + B_{\alpha \gamma k} D_k u_\gamma + C_{\alpha \delta} u_\delta = f_\alpha,$$

can be compactly written as  $A : D^2u + B : Du + Cu = f$ , where the meaning of “:” in the respective dimensions is made clear by the context. Let now  $P : \mathbb{R}^n \rightarrow \mathbb{R}^N$  be linear map. The identity map of  $\mathbb{R}^N$  splits as  $I = [P]^\top \oplus [P]^\perp$ , where  $[P]^\top$  and  $[P]^\perp$  denote orthogonal projection on range  $R(P)$  and nullspace  $N(P^\top)$  respectively. Moreover, for the dilation function (1.3), we have  $K(P) \geq n$  and  $K(P) = n$  if and only if  $P^\top P = \lambda I$  with  $\lambda = \frac{1}{n} |P|^2$ . This property of  $K$  is a simple consequence of the inequality of arithmetic-geometric mean applied to the  $n$  eigenvalues of  $P^\top P$  and an application of the Spectral Theorem. Let us now recall some elementary properties of determinants. If  $A = A_{ij} e_i \otimes e_j \in \mathbb{R}^n \otimes \mathbb{R}^n$ , we have

$$(2.3) \quad \text{cof}(A)_{ij} := (-1)^{i+j} \det \left( \sum_{k \neq i, l \neq j} A_{kl} e_k \otimes e_l \right),$$

$$(2.4) \quad \text{cof}(A) := \text{cof}(A)_{ij} e_i \otimes e_j,$$

$$(2.5) \quad A \operatorname{cof}(A)^\top = \operatorname{cof}(A)^\top A = \det(A)I,$$

$$(2.6) \quad D_{A_{ij}}(\det(A)) \equiv (\det(A))_{A_{ij}} = \operatorname{cof}(A)_{ij}.$$

Obviously, subscript denotes partial derivative. The *Ahlfors operator* is defined by

$$(2.7) \quad S(A) := \frac{1}{2}(A + A^\top) - \frac{1}{n}\operatorname{tr}(A)I$$

and has the property that for any  $A$ ,  $S(A)$  is symmetric and traceless, that is  $\operatorname{tr}(S(A)) = 0$ . Let now  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^1(\Omega)^N$ . Then, the rank of  $Du$  satisfies  $\operatorname{rk}(Du) = n \leq N$ .  $u$  is *Conformal* when there is  $f \in C^0(\Omega)$  such that  $Du^\top Du = f^2 I$  on  $\Omega$ , that is  $D_i u_\alpha D_j u_\alpha = f^2 \delta_{ij}$ . For immersions, the Riemannian metric on  $u(\Omega)$  induced from  $\mathbb{R}^N$  is  $g := Du^\top Du$  and  $g^{-1}$  denotes the pointwise inverse of the positive symmetric tensor  $g$ . Since  $S(g) = g - \frac{1}{n}\operatorname{tr}(g)I$ , we have the commutativity relation

$$(2.8) \quad g^{-1}S(g) = S(g)g^{-1} = I - \frac{\operatorname{tr}(g)}{n}g^{-1}$$

which will be tacitly used in the sequel. In view of these conventions, the Aronsson system describing Extremal Quasiconformal immersions in index form reads

$$(2.9) \quad \left( K_{P_{\alpha i}} K_{P_{\beta j}} + [K_P]_{\alpha\gamma}^\perp K_{P_{\gamma i} P_{\beta j}} \right) (Du) D_{ij}^2 u_{\beta} = 0.$$

The derivatives  $K_P, K_{PP}$  of  $K$  appearing here and in (1.10), (1.11) are given in index form by (3.2), (3.10). Finally, we will use the notation “ $\Gamma$ ” for sections of vector bundles. We note that our terminology of “ $p$ -Quasiconformal” is in slight conflict with the usage of this term in the literature, but is used in order to avoid the less elegant term “ $L^p$ -Quasiconformal”. Since we are only interested in the extreme case of  $p = \infty$ , there will be no confusion. We conclude by observing that when  $\Omega \subset \subset \mathbb{R}^n$ , all immersions  $u : \bar{\Omega} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  in  $C^1(\bar{\Omega})^N$  are  $p$ -Quasiconformal for all  $p \in [1, \infty]$ .

### 3. DERIVATION OF THE EULER-LAGRANGE PDE SYSTEM GOVERNING EXTREMAL $p$ -QUASICONFORMAL IMMERSIONS.

In this section we calculate the specific form of the Euler-Lagrange system associated to the functional  $\|K(Du)\|_{L^p(\Omega)}^p$  which Extremal  $p$ -Quasiconformal immersions satisfy. We begin by calculating first and second derivatives of (1.3).

**Lemma 3.1.** *Let  $K$  be given by (1.3). Then,  $K \in C^1(S^+)$  and its derivative is given by*

$$(3.1) \quad K_P(P) = 2P \frac{(P^\top P)^{-1} S(P^\top P)}{\det(P^\top P)^{1/n}}$$

which in index form is

$$(3.2) \quad K_{P_{\alpha i}}(P) = 2P_{\alpha m} \left( \frac{\delta_{mi} - \frac{1}{n}|P|^2 (P^\top P)^{-1}_{mi}}{\det(P^\top P)^{1/n}} \right).$$

**Proof of Lemma 3.1.** We begin by observing the triviality that for  $P \in S^+$ , the matrix  $P^\top P$  is positive symmetric on  $\mathbb{R}^n$  and also

$$(3.3) \quad (P^\top P)^{-1, \top} = (P^\top P)^{\top, -1} = (P^\top P)^{-1}.$$



By differentiation of (1.3), we have

$$\begin{aligned}
 K_{P_{\alpha i}}(P) &= \frac{2P_{\alpha i} \det(P^\top P)^{\frac{1}{n}} - \frac{|P|^2}{n} \det(P^\top P)^{\frac{1}{n}-1} \operatorname{cof}(P^\top P)_{kl} (P_{\beta k} P_{\beta l})_{P_{\alpha i}}}{\det(P^\top P)^{2/n}} \\
 (3.4) \quad &= \frac{2P_{\alpha i} - \frac{|P|^2}{n \det(P^\top P)} \operatorname{cof}(P^\top P)_{kl} (\delta_{\alpha\beta} \delta_{ik} P_{\beta l} + \delta_{\alpha\beta} \delta_{il} P_{\beta k})}{\det(P^\top P)^{1/n}}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 K_{P_{\alpha i}}(P) &= \frac{2P_{\alpha i} - \frac{|P|^2}{n \det(P^\top P)} (\operatorname{cof}(P^\top P)_{il} P_{\alpha l} + \operatorname{cof}(P^\top P)_{ki} P_{\alpha k})}{\det(P^\top P)^{1/n}} \\
 (3.5) \quad &= 2P_{\alpha m} \frac{\delta_{mi} - \frac{|P|^2}{n \det(P^\top P)} \frac{1}{2} (\operatorname{cof}(P^\top P)_{im} + \operatorname{cof}(P^\top P)_{mi})}{\det(P^\top P)^{1/n}}.
 \end{aligned}$$

Hence, (3.5) gives

$$(3.6) \quad K_P(P) = \frac{2P}{\det(P^\top P)^{1/n}} \left( I - \frac{|P|^2}{n} \left( \frac{\operatorname{cof}(P^\top P)^\top + \operatorname{cof}(P^\top P)}{2 \det(P^\top P)} \right) \right)$$

and by using that

$$(3.7) \quad \operatorname{cof}(P^\top P)^\top = \operatorname{cof}(P^\top P) = (P^\top P)^{-1} \det(P^\top P),$$

equation (3.6) gives

$$\begin{aligned}
 K_P(P) &= \frac{2P}{\det(P^\top P)^{1/n}} \left( I - \frac{|P|^2}{n} (P^\top P)^{-1} \right) \\
 (3.8) \quad &= 2P \frac{(P^\top P)^{-1}}{\det(P^\top P)^{1/n}} \left( P^\top P - \frac{|P|^2}{n} I \right).
 \end{aligned}$$

In view of (3.8), formula (3.1) has been established.  $\square$

**Lemma 3.2.** *Let  $K$  be given by (1.3). Then,  $K \in C^2(S^+)$  and its 2nd derivative is given by*

$$(3.9) \quad K_{PP}(P) = 2I \otimes \frac{(P^\top P)^{-1} S(P^\top P)}{\det(P^\top P)^{1/n}} + 2P \otimes P : \frac{(P^\top P)^{-1} E}{\det(P^\top P)^{1/n}} + O(P)$$

which in index form is

$$\begin{aligned}
 K_{P_{\alpha i} P_{\beta j}}(P) &= 2\delta_{\alpha\beta} \left( \frac{(P^\top P)_{ik}^{-1} (P_{\gamma k} P_{\gamma j} - \frac{1}{n} |P|^2 \delta_{kj})}{\det(P^\top P)^{1/n}} \right) \\
 (3.10) \quad &+ 2P_{\alpha m} P_{\beta l} \left( \frac{(P^\top P)_{ik}^{-1} E_{kjl m}}{\det(P^\top P)^{1/n}} \right) + O_{\alpha i \beta j}(P).
 \end{aligned}$$



Here  $O_{\alpha i \beta j}(P)$  is a tensor of the form  $K_{P_{\alpha m}}(P)A_{m \beta i j}(P)$  and is annihilated by  $[K_P(P)]_{\gamma \alpha}^\perp$ , that is  $[K_P(P)]^\perp O(P) = 0$ .  $E$  denotes a constant 4-tensor whose components  $E_{k j l m}$  are given by

$$(3.11) \quad E_{k j l m} := \delta_{m l} \delta_{j k} + \delta_{m j} \delta_{k l} - \frac{2}{n} \delta_{m k} \delta_{j l}.$$

The explicit form of the tensor  $O_{\alpha i \beta j}(P)$  is a complicated formula which follows by the proof of Lemma 3.2, but we do not need this formula because is “killed” by  $[K_P(P)]^\perp$  and does not appear in (1.9).

**Proof of Lemma 3.2.** We begin by calculating the derivative  $((P^\top P)_{mi}^{-1})_{P_{\beta j}}$ . We have

$$(3.12) \quad (P^\top P)_{mi}^{-1} (P^\top P)_{ik} = \delta_{mk}$$

which gives

$$(3.13) \quad \begin{aligned} ((P^\top P)_{mi}^{-1})_{P_{\beta j}} (P^\top P)_{ik} &= -(P^\top P)_{mi}^{-1} (P_{\gamma i} P_{\gamma k})_{P_{\beta j}} \\ &= -(P^\top P)_{mi}^{-1} [\delta_{\beta \gamma} \delta_{ij} P_{\gamma k} + P_{\gamma i} \delta_{\beta \gamma} \delta_{kj}] \\ &= -(P^\top P)_{ml}^{-1} [P_{\beta k} \delta_{lj} + P_{\beta l} \delta_{kj}]. \end{aligned}$$

Hence, we have

$$(3.14) \quad ((P^\top P)_{mi}^{-1})_{P_{\beta j}} = -(P^\top P)_{ml}^{-1} [P_{\beta k} \delta_{lj} + P_{\beta l} \delta_{kj}] (P^\top P)_{ki}^{-1}.$$

Now we differentiate (3.2):

$$(3.15) \quad \begin{aligned} K_{P_{\alpha i} P_{\beta j}}(P) &= 2\delta_{\alpha \beta} \delta_{mj} \left( \frac{\delta_{mi} - \frac{1}{n} |P|^2 (P^\top P)_{mi}^{-1}}{\det(P^\top P)^{1/n}} \right) - 2P_{\alpha m} \left( \frac{(|P|^2 (P^\top P)_{mi}^{-1})_{P_{\beta j}}}{n \det(P^\top P)^{1/n}} \right) \\ &\quad - \left[ 2P_{\alpha m} \left( \frac{\delta_{mi} - \frac{1}{n} |P|^2 (P^\top P)_{mi}^{-1}}{\det(P^\top P)^{1/n}} \right) \right] \frac{(\det(P^\top P)^{1/n})_{P_{\beta j}}}{\det(P^\top P)^{1/n}}. \end{aligned}$$

In view of (3.2), the last summand in (3.15) is annihilated by the projection  $[K_P(P)]_{\gamma \alpha}^\perp$ . We rewrite (3.15) as

$$(3.16) \quad \begin{aligned} K_{P_{\alpha i} P_{\beta j}}(P) &= 2\delta_{\alpha \beta} \left( \frac{\delta_{ij} - \frac{1}{n} |P|^2 (P^\top P)_{ij}^{-1}}{\det(P^\top P)^{1/n}} \right) \\ &\quad - 2P_{\alpha m} \left( \frac{(|P|^2 (P^\top P)_{mi}^{-1})_{P_{\beta j}}}{n \det(P^\top P)^{1/n}} \right) + O_{\alpha i \beta j}(P). \end{aligned}$$

By using (3.14) in (3.16), we have

$$(3.17) \quad K_{P_{\alpha i} P_{\beta j}}(P) = 2\delta_{\alpha \beta} \left( \frac{\delta_{ij} - \frac{1}{n} |P|^2 (P^\top P)_{ij}^{-1}}{\det(P^\top P)^{1/n}} \right) + S_{\alpha i \beta j}(P) + O_{\alpha i \beta j}(P),$$

where we have set

$$(3.18) \quad S_{\alpha i \beta j}(P) := \frac{2}{n} P_{\alpha m} \frac{2P_{\beta j} (P^\top P)_{mi}^{-1} - |P|^2 (P^\top P)_{ml}^{-1} [P_{\beta k} \delta_{lj} + P_{\beta l} \delta_{kj}] (P^\top P)_{ki}^{-1}}{\det(P^\top P)^{1/n}}.$$

Equation (3.18) gives

$$\begin{aligned}
 S_{\alpha i \beta j}(P) = & -\frac{4}{n} P_{\alpha m} P_{\beta j} \frac{(P^\top P)_{mi}^{-1}}{\det(P^\top P)^{1/n}} \\
 & + 2P_{\alpha m} \left( \frac{\frac{1}{n}|P|^2 (P^\top P)_{mj}^{-1}}{\det(P^\top P)^{1/n}} \right) (P^\top P)_{ki}^{-1} P_{\beta k} \\
 & + 2P_{\alpha m} \left( \frac{\frac{1}{n}|P|^2 (P^\top P)_{mk}^{-1}}{\det(P^\top P)^{1/n}} \right) (P^\top P)_{ij}^{-1} P_{\beta k}.
 \end{aligned}
 \tag{3.19}$$

We rewrite (3.19) as

$$\begin{aligned}
 S_{\alpha i \beta j}(P) = & -\frac{4}{n} P_{\alpha m} P_{\beta j} \frac{(P^\top P)_{mi}^{-1}}{\det(P^\top P)^{1/n}} \\
 & + 2P_{\alpha m} \left( \frac{-\delta_{mj} + \frac{1}{n}|P|^2 (P^\top P)_{mj}^{-1}}{\det(P^\top P)^{1/n}} + \frac{\delta_{mj}}{\det(P^\top P)^{1/n}} \right) (P^\top P)_{ki}^{-1} P_{\beta k} \\
 & + 2P_{\alpha m} \left( \frac{-\delta_{mk} + \frac{1}{n}|P|^2 (P^\top P)_{mk}^{-1}}{\det(P^\top P)^{1/n}} + \frac{\delta_{mk}}{\det(P^\top P)^{1/n}} \right) (P^\top P)_{ij}^{-1} P_{\beta k}
 \end{aligned}
 \tag{3.20}$$

and observe that in view of (3.2),  $[K_P(Du)]_{\gamma\alpha}^\perp$  annihilates the first summands in the brackets of (3.20) and  $S_{\alpha i \beta j}(P)$  simplifies to

$$\begin{aligned}
 S_{\alpha i \beta j}(P) = & 2 \frac{P_{\alpha k} P_{\beta k} (P^\top P)_{ij}^{-1} + P_{\alpha j} P_{\beta k} (P^\top P)_{ki}^{-1} - \frac{2}{n} P_{\alpha m} P_{\beta j} (P^\top P)_{mi}^{-1}}{\det(P^\top P)^{1/n}} \\
 & + O_{\alpha i \beta j}(P),
 \end{aligned}
 \tag{3.21}$$

for some tensor  $O_{\alpha i \beta j}(P)$  annihilated by  $[K_P(Du)]_{\gamma\alpha}^\perp$ . We rewrite (3.21) as

$$S_{\alpha i \beta j}(P) = 2P_{\alpha m} P_{\beta l} (P^\top P)_{ki}^{-1} \left( \frac{\delta_{ml} \delta_{jk} + \delta_{mj} \delta_{kl} - \frac{2}{n} \delta_{mk} \delta_{jl}}{\det(P^\top P)^{1/n}} \right) + O_{\alpha i \beta j}(P).
 \tag{3.22}$$

In view of (3.22), (3.18), (3.17) and (3.11), equation (3.10) follows.  $\square$

In view of Lemma 3.1, the Euler-Lagrange system describing  $p$ -Quasiconformal immersions  $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$  is

$$Q_p u := \operatorname{Div} \left( K(Du)^{p-1} K_P(Du) \right) = 0.
 \tag{3.23}$$

In view of (3.1), (3.23) can be written in index form as

$$D_i \left( \left( \frac{\operatorname{tr}(g)}{\det(g)^{1/n}} \right)^{p-1} D_k u_\alpha \frac{g_{km}^{-1} S(g)_{mi}}{\det(g)^{1/n}} \right) = 0,
 \tag{3.24}$$

where  $g = Du^\top Du$  is the Riemannian metric and  $S$  is the Ahlfors operator of (2.7).

#### 4. DERIVATION OF THE ARONSSON PDE SYSTEM GOVERNING EXTREMAL $\infty$ -QUASICONFORMAL IMMERSIONS.

The derivation we perform in this section can be deduced by a reworking of our results in [K1, K2] and application of Lemmas 3.1 and 3.2 proved previously, but for the reader's convenience it is best to argue at the outset. Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . By distributing derivatives in (3.23), we have

$$(4.1) \quad (p-1)K^{p-2}K_{P_{\alpha i}}(Du)K_{P_{\beta j}}(Du)D_{ij}^2 u_{\beta} + K^{p-1}K_{P_{\alpha i}P_{\beta j}}(Du)D_{ij}^2 u_{\beta} = 0.$$

For each  $x \in \Omega$ ,  $K_P((Du)(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^N$  is a linear map. We define the orthogonal projections

$$(4.2) \quad [K_P(Du)]^{\perp} := \text{Proj}_{N((K_P(Du))^{\top})},$$

$$(4.3) \quad [K_P(Du)]^{\top} := \text{Proj}_{R(K_P(Du))},$$

which are the projections on nullspace of  $(K_P(Du))^{\top}$  and range of  $K_P(Du)$  respectively. We rewrite (4.1) by applying the expansion  $I = [K_P(Du)]^{\perp} + [K_P(Du)]^{\top}$  of the identity of  $\mathbb{R}^N$  and contract the derivative in the left hand side to obtain

$$(4.4) \quad \begin{aligned} & K_P(Du)D(K(Du)) + \frac{K}{p-1}[K_P(Du)]^{\top}K_{PP}(Du) : D^2u \\ &= -\frac{K}{p-1}[K_P(Du)]^{\perp}K_{PP}(Du) : D^2u. \end{aligned}$$

The left hand side is a vector valued in  $[K_P(Du)]^{\top}$  and the right hand side is a vector valued in  $[K_P(Du)]^{\perp}$ . By orthogonality, left and right hand side vanish and actually (4.4) decouples to two systems. We rescale the right hand side of (4.4) by multiplying by  $(p-1)K(Du)^{-1}$  which is possible since  $K(Du) \geq n > 0$  and rearrange to obtain

$$(4.5) \quad \begin{aligned} & K_P(Du) \otimes K_P(Du) : D^2u + [K_P(Du)]^{\perp}K_{PP}(Du) : D^2u \\ &= -\frac{K(Du)}{p-1}[K_P(Du)]^{\top}K_{PP}(Du) : D^2u. \end{aligned}$$

We rewrite as

$$(4.6) \quad \left( K_P \otimes K_P + [K_P]^{\perp}K_{PP} \right)(Du) : D^2u = -\frac{K[K_P]^{\top}K_{PP}}{p-1}(Du) : D^2u.$$

As  $p \rightarrow \infty$ , (4.6) leads to (1.9).

**Remark 4.1.** We note that we can remove the dilation function  $K$  from the normal coefficient  $[K_P]^{\perp}K_{PP}$  because it is strictly positive. We do not have this option in the case of the general Aronsson system (1.7), because  $|H(Du)|$  may vanish. However, when  $n = 2 \leq N$  and  $H(P) = |P|^2$ , in [K2] we showed that non-constant  $\infty$ -Harmonic maps have no interior gradient zeros: either  $|Du| > 0$  or  $|Du| \equiv 0$ .

The next differential identity relates our system (1.9) with the seemingly different Aronsson PDE system of Capogna-Raich in [CR]. In particular, it follows that even when  $n = N$  the PDE system derived in [CR] is only a projection of (1.9) along  $[K_P(Du)]^{\top}$ . Hence, the PDE system in [CR] fails to encapsulate all the information of extremal quasiconformality.

**Lemma 4.2.** *Let  $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$  be a local diffeomorphism in  $C^1(\Omega)^n$ . Then, we have the identity*

$$(4.7) \quad K_P(Du) = -\frac{2K(Du)}{n} \left( (Du)^{-1, \top} - n \frac{Du}{|Du|^2} \right)$$

where  $K$  and  $K_P$  are given by (1.3) and (3.1).

**Proof of Lemma 4.2.** By observing that for any invertible  $A \in \mathbb{R}^n \otimes \mathbb{R}^n$  there holds  $A^{-1, \top} = A^{\top, -1}$ , we have

$$(4.8) \quad (Du^{\top} Du)^{-1} = (Du)^{-1} (Du)^{\top, -1} = (Du)^{-1} (Du)^{-1, \top}.$$

Thus, we obtain

$$(4.9) \quad \begin{aligned} (Du)^{-1, \top} - n \frac{Du}{|Du|^2} &= -\frac{n}{|Du|^2} \left( Du - \frac{|Du|^2}{n} (Du)^{-1, \top} \right) \\ &= -\frac{n}{|Du|^2} \left( Du - \frac{|Du|^2}{n} Du (Du)^{-1} (Du)^{-1, \top} \right) \\ &= -\frac{n}{|Du|^2} Du \left( I - \frac{|Du|^2}{n} (Du)^{-1} (Du)^{-1, \top} \right). \end{aligned}$$

Consequently, by (4.8) and (4.9), we obtain

$$(4.10) \quad \begin{aligned} -\frac{|Du|^2}{n} \left( (Du)^{-1, \top} - n \frac{Du}{|Du|^2} \right) &= Du \left( I - \frac{|Du|^2}{n} (Du^{\top} Du)^{-1} \right) \\ &= Du (Du^{\top} Du)^{-1} \left( Du^{\top} Du - \frac{|Du|^2}{n} I \right). \end{aligned}$$

Hence, by (3.1) and (1.3) we have

$$(4.11) \quad \begin{aligned} -\frac{2K(Du)}{n} \left( (Du)^{-1, \top} - n \frac{Du}{|Du|^2} \right) &= 2Du (Du^{\top} Du)^{-1} \left( \frac{Du^{\top} Du - \frac{|Du|^2}{n} I}{\det(Du^{\top} Du)^{1/n}} \right) \\ &= K_P(Du). \end{aligned}$$

The desired identity follows.  $\square$

## 5. VARIATIONAL STRUCTURE OF EXTREMAL $\infty$ -QUASICONFORMAL IMMERSIONS.

We begin by introducing a minimality notion of vector-valued Calculus of Variations in  $L^\infty$  for the supremal dilation functional (1.2). Let  $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$  be an immersion in  $C^1(\Omega)^N$ . In view of (3.1), we have the identity

$$(5.1) \quad K_P(Du) = \left( 2 \frac{Du (Du^{\top} Du)^{-1}}{\det(Du^{\top} Du)^{1/n}} \right) S(Du^{\top} Du).$$

Generally, the rank of  $K_P(Du)$  may not be constant throughout  $\Omega$ , although by assumption  $\text{rk}(Du) = \text{rk}(Du^{\top} Du) \equiv n$ , because possibly  $\text{rk}(S(Du^{\top} Du)) < n$  on certain regions of  $\Omega$ . We set

$$(5.2) \quad \Omega_k := \text{int} \left\{ \text{rk}(S(Du^{\top} Du)) = k \right\}, \quad k = 0, 1, \dots, n,$$

where “int” denotes topological interior. The  $n + 1$  open sets  $\Omega_k$  are the “*phases*” of the immersion  $u$ . Their complement in  $\Omega$

$$(5.3) \quad \mathcal{S} := \Omega \setminus (\cup_0^n \Omega_k)$$

is the set of “*interfaces*” and is closed in  $\Omega$  with empty interior. We will also need the “*augmented phases*”

$$(5.4) \quad \Omega_k^* := \left\{ \text{rk}(S(Du^\top Du)) = k \right\}, \quad k = 0, 1, \dots, n.$$

Obviously,  $\{\Omega_0^*, \dots, \Omega_n^*\}$  is a partition of  $\Omega$  to disjoint phases and  $\mathcal{S}$  can be written as  $\mathcal{S} = \cup_0^n (\Omega_k^* \setminus \Omega_k)$ . The extreme cases of  $\Omega_0^*$  and  $\Omega_n^*$  are particularly important.  $\Omega_0^*$  is the *conformality set* of the immersion and is closed in  $\Omega$ . Hence,

$$(5.5) \quad \Omega_0^* = \left\{ Du^\top Du = \frac{|Du|^2}{n} I \right\}.$$

Similarly, by Corollary 6.3 that follows, if  $u$  solves  $K_P(Du) \otimes K_P(Du) : D^2u = 0$ , then  $\Omega_n^*$  is the *constant dilation set* of the immersion and coincides with  $\Omega_n$ :

$$(5.6) \quad \Omega_n^* = \left\{ \frac{|Du|^2}{\det(Du^\top Du)^{1/n}} = \text{const.} \right\}.$$

If  $\Omega_n$  is not connected, then the constants may differ in connected components.

**Definition 5.1.** Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^1(\Omega)^N$ .

(i) We say that  $u$  has *Rank-One Locally Minimal Dilation* when for all compactly contained subdomains  $D$  of  $\Omega$ , all functions  $g$  over  $D$  vanishing on  $\partial D$  and all directions  $\xi$ ,  $u$  is a minimizer on  $D$  with respect to essentially scalar variations  $u + f\xi$ :

$$(5.7) \quad \left. \begin{array}{l} D \subset\subset \Omega, \\ f \in C_0^1(D), \\ \xi \in \mathbb{S}^{N-1} \end{array} \right\} \implies K_\infty(u, \Omega) \leq K_\infty(u + f\xi, \Omega).$$

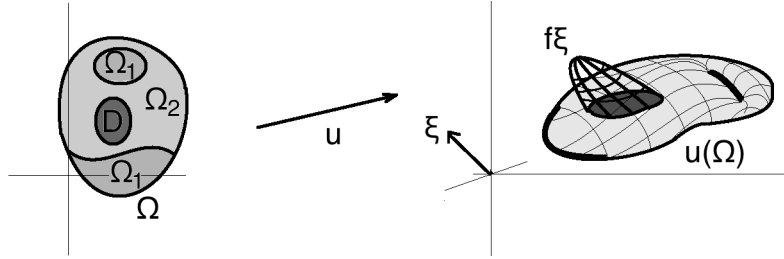


Figure 2.

(ii) We say that  $u(\Omega)$  has *Minimally Distorted Area* when for all compactly contained subdomains  $D$  off the interfaces, all functions  $h$  on  $\bar{D}$  (not only vanishing on  $\partial D$ ) and all vector fields  $\nu$  along  $u$  normal to  $K_P(Du)$ ,  $u$  is a minimizer on  $D$  with respect to normal free variations  $u + h\nu$ :

$$(5.8) \quad \left. \begin{array}{l} D \subset\subset \Omega \setminus \mathcal{S}, \\ h \in C^1(\bar{D}), \\ \nu \in \Gamma([K_P(Du)]^\perp) \end{array} \right\} \implies K_\infty(u, \Omega) \leq K_\infty(u + h\nu, \Omega).$$

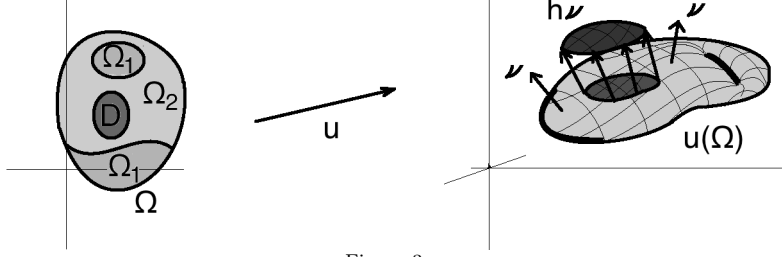


Figure 3.

(iii) We call  $u$  *Minimal  $\infty$ -Quasiconformal Immersion* when  $u$  has Rank-One Locally Minimal Dilation with Minimally Distorted Area of  $u(\Omega) \subseteq \mathbb{R}^N$ .

By employing the previous minimality notion, we have the next

**Theorem 5.2** (Variational Structure of Extremal  $\infty$ -Quasiconformal Immersions). *Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . Then, if  $u$  is Minimal  $\infty$ -Quasiconformal, it follows that  $u$  solves*

$$(5.9) \quad K_P(Du) \otimes K_P(Du) : D^2u = 0, \quad \text{on } \Omega,$$

$$(5.10) \quad [K_P(Du)]^\perp K_{PP}(Du) : D^2u = 0, \quad \text{on } \Omega \setminus \mathcal{S},$$

where  $\mathcal{S}$  is the set of interfaces of rank discontinuities of  $S(Du^\top Du)$ .

We note that by the results of Section 6 that follows, in the case  $n = 2 \leq N$  Theorem 5.2 can be strengthened to the following

**Corollary 5.3** (2-Dimensional Extremal  $\infty$ -Quasiconformal Immersions). *Let  $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . If  $u$  is Minimal  $\infty$ -Quasiconformal, it follows that  $u$  is Extremal  $\infty$ -Quasiconformal.*

The point in Corollary 5.3 is that (5.10) is satisfied on  $\Omega$  and not only on  $\Omega \setminus \mathcal{S}$ . Actually, when  $n = 2$  then the set of interfaces is empty:  $\mathcal{S} = \emptyset$ .

The proof of Theorem 5.2 is split in two lemmas.

**Lemma 5.4.** *Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . If  $u$  has Rank-One Locally Minimal Dilation, then  $u$  solves  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  on  $\Omega$ .*

The proof of Lemma 5.4 follows by Theorem 2.1 in [K1] and relates to Lemma 2.3 in [K3], but we present a simplified more direct proof for the reader's convenience.

**Proof of Lemma 5.4.** Fix  $x \in \Omega$ ,  $0 < \varepsilon < \text{dist}(x, \partial\Omega)$ ,  $\delta > 0$  and  $\xi \in \mathbb{S}^{N-1}$ . Choose  $D := \mathbb{B}_\varepsilon(x)$  and  $f \in C_0^1(D)$  given by

$$(5.11) \quad f(z) := \frac{1}{2}(\varepsilon^2 - |z - x|^2).$$

Since  $\text{rk}(Du) = n$  on  $\Omega$  and  $Df(z) = -(z - x)$ , by restricting  $\delta$  sufficiently we obtain that  $\text{rk}(Du + \delta\xi \otimes Df) = n$  on  $\mathbb{B}_\varepsilon(x)$ . By Taylor expansions of  $K(Du)$  and  $K(Du + \delta\xi \otimes Df)$  at  $x$  we have

$$(5.12) \quad K(Du(z)) = K(Du(x)) + D(K(Du))(x)^\top (z - x) + o(|z - x|),$$

as  $z \rightarrow x$ , and also by using that  $D^2f = -I$  and  $Df(x) = 0$  we have

$$\begin{aligned}
 K((Du + \delta\xi \otimes Df)(z)) &= K((Du + \delta\xi \otimes Df)(x)) \\
 &\quad + D(K(Du + \delta\xi \otimes Df))(x)^\top(z - x) + o(|z - x|) \\
 (5.13) \quad &= K(Du(x)) + K_P(Du(x))^\top(D^2u(x) - \delta\xi \otimes I)(z - x) \\
 &\quad + o(|z - x|) \\
 &= K(Du(x)) + \left(D(K(Du))^\top - \delta\xi^\top K_P(Du)\right)(x)(z - x) \\
 &\quad + o(|z - x|),
 \end{aligned}$$

as  $z \rightarrow x$ . By (5.12) we have the estimate

$$\begin{aligned}
 K_\infty(u, \mathbb{B}_\varepsilon(x)) &\geq K(Du(x)) + \max_{\{|z-x| \leq \varepsilon\}} \left\{ D(K(Du))(x)^\top(z - x) \right\} + o(\varepsilon) \\
 (5.14) \quad &= K(Du(x)) + \varepsilon |D(K(Du))(x)| + o(\varepsilon),
 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , and also by (5.13) we have

$$\begin{aligned}
 K_\infty(u + \delta f\xi, \mathbb{B}_\varepsilon(x)) &\leq K(Du(x)) + \max_{\{|z-x| \leq \varepsilon\}} \left\{ D(K(Du))^\top \right. \\
 (5.15) \quad &\quad \left. - \delta\xi^\top K_P(Du)(x)(z - x) \right\} + o(\varepsilon) \\
 &= K(Du(x)) + \varepsilon |D(K(Du)) - \delta\xi^\top K_P(Du)|(x) + o(\varepsilon),
 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Then, since  $u$  has rank-one locally minimal dilation, by (5.14) and (5.15) we have

$$\begin{aligned}
 0 &\leq K_\infty(u + \delta f\xi, \mathbb{B}_\varepsilon(x)) - K_\infty(u, \mathbb{B}_\varepsilon(x)) \\
 (5.16) \quad &\leq \varepsilon \left( |D(K(Du)) - \delta\xi^\top K_P(Du)| - |D(K(Du))| \right)(x) + o(\varepsilon),
 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Suppose first  $D(K(Du))(x) = 0$ . Since

$$(5.17) \quad K_P(Du) \otimes K_P(Du) : D^2u = K_P(Du)D(K(Du))$$

we obtain that  $(K_P(Du) \otimes K_P(Du) : D^2u)(x) = 0$  as desired. If  $D(K(Du))(x) \neq 0$ , then Taylor expansion of the function

$$(5.18) \quad p \mapsto |D(K(Du))(x) + p| - |D(K(Du))(x)|$$

at  $p_0 = 0$  and evaluated at  $p = -\delta\xi^\top K_P(Du(x))$ , (5.16) implies after letting  $\varepsilon \rightarrow 0$  that

$$(5.19) \quad 0 \leq -\delta\xi^\top K_P(Du(x)) \left( \frac{D(K(Du))}{|D(K(Du))|} \right)(x) + o(\delta).$$

By letting  $\delta \rightarrow 0$  in (5.19) we obtain  $\xi^\top (K_P(Du) \otimes K_P(Du) : D^2u)(x) \geq 0$  for any direction  $\xi$ . Since  $\xi$  and  $x$  are arbitrary we get  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  on  $\Omega$ . The lemma follows.  $\square$

**Lemma 5.5.** *Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$  with Minimally Distorted Area of  $u(\Omega)$ . Then,  $u$  solves  $[K_P(Du)]^\perp K_{PP}(Du) : D^2u = 0$  on  $\Omega \setminus \mathcal{S}$ .*

**Proof of Lemma 5.5.** Fix  $x \in \Omega \setminus \mathcal{S}$ . Then,  $x$  belongs to some phase  $\Omega_k$  of constant rank and  $\text{rk}(S(Du^\top Du)) \equiv k$  thereon. We choose  $0 < \varepsilon < \frac{1}{2}\text{dist}(x, \partial\Omega_k)$



and  $0 < \delta < 1$ . By the Rank Theorem (see e.g. [N]) and application of the Gram-Schmidt procedure to a local frame field adapted to the immersion near  $u(x)$ , we can construct a local frame of sections  $\{\nu^1, \dots, \nu^{N-k}\}$  spanning  $\Gamma([K_P(Du)]^\perp, \mathbb{B}_{2\varepsilon}(x))$  for  $\varepsilon$  small enough. Let  $\nu$  be a linear combination of these sections and choose an  $h \in C^1(\overline{\mathbb{B}_\varepsilon(x)})$ . Since  $\text{rk}(Du) = n$  on  $\Omega$ , by restricting  $\delta$  sufficiently we obtain  $\text{rk}(D(u + \delta h\nu)) = n$  on  $\mathbb{B}_\varepsilon(x)$ . By differentiating  $\nu^\top K_P(Du) = 0$  we obtain

$$(5.20) \quad D_k \nu_\alpha K_{P_{\alpha i}}(Du) = -\nu_\alpha K_{P_{\alpha i} P_{\beta j}}(Du) D_{kj}^2 u_\beta$$

and by putting  $i = k$  and summing, we get

$$(5.21) \quad D_i \nu_\alpha K_{P_{\alpha i}}(Du) = -\nu_\alpha K_{P_{\alpha i} P_{\beta j}}(Du) D_{ij}^2 u_\beta$$

that is

$$(5.22) \quad D\nu : K_P(Du) = -\nu^\top K_{PP}(Du) : D^2 u.$$

By Taylor expansion of the dilation and usage of  $\nu^\top K_P(Du) = 0$ , we obtain

$$(5.23) \quad \begin{aligned} K(D(u + \delta h\nu)) &= K(Du) + K_P(Du) : D(\delta h\nu) + o(\delta|h\nu|) \\ &= K(Du) + \delta K_P(Du) : (hD\nu + \nu \otimes Dh) + o(\delta) \\ &= K(Du) + \delta(hD\nu : K_P(Du) + \nu^\top K_P(Du) Dh) + o(\delta) \\ &= K(Du) + \delta hD\nu : K_P(Du) + o(\delta) \end{aligned}$$

as  $\delta \rightarrow 0$ . By (5.23) and (5.22) we have

$$(5.24) \quad K(D(u + \delta h\nu)) = K(Du) - 2\delta h(\nu^\top K_{PP}(Du) : D^2 u) + o(\delta),$$

as  $\delta \rightarrow 0$ . Hence, since  $u(\Omega)$  has minimally distorted area, by (5.24) we have

$$(5.25) \quad \begin{aligned} K_\infty(u, \mathbb{B}_\varepsilon(x)) &\leq K_\infty(u + \delta h\nu, \mathbb{B}_\varepsilon(x)) \\ &= \sup_{\mathbb{B}_\varepsilon(x)} \left\{ K(Du) - 2\delta h(\nu^\top K_{PP}(Du) : D^2 u) + o(\delta) \right\} \end{aligned}$$

as  $\delta \rightarrow 0$ , which gives

$$(5.26) \quad \begin{aligned} K_\infty(u, \mathbb{B}_\varepsilon(x)) &\leq \sup_{\mathbb{B}_\varepsilon(x)} K(Du) - 2\delta \min_{\mathbb{B}_\varepsilon(x)} \left\{ h(\nu^\top K_{PP}(Du) : D^2 u) \right\} + o(\delta) \\ &= K_\infty(u, \mathbb{B}_\varepsilon(x)) - 2\delta \min_{\mathbb{B}_\varepsilon(x)} \left\{ h(\nu^\top K_{PP}(Du) : D^2 u) \right\} + o(\delta). \end{aligned}$$

Hence, by passing to the limit as  $\delta \rightarrow 0$ , (5.26) gives

$$(5.27) \quad \min_{\mathbb{B}_\varepsilon(x)} \left\{ h(\nu^\top K_{PP}(Du) : D^2 u) \right\} \leq 0.$$

We now choose as  $h$  the constant function

$$(5.28) \quad h := \text{sgn}(\nu^\top K_{PP}(Du) : D^2 u)(x)$$

and by (5.27) as  $\varepsilon \rightarrow 0$  we get  $|\nu^\top K_{PP}(Du) : D^2 u|(x) = 0$ . Since  $\nu$  is an arbitrary normal section and  $x$  is an arbitrary point on  $\Omega \setminus \mathcal{S}$ , we get  $([K_P]^\perp K_{PP})(Du) : D^2 u = 0$  on  $\Omega \setminus \mathcal{S}$  and the lemma follows.  $\square$

6. GEOMETRY OF EXTREMAL  $\infty$ -QUASICONFORMAL IMMERSIONS.

**6.1. Geometric Form of Aronsson PDE System Describing Extremal  $\infty$ -Quasiconformal Immersions.** In this subsection we show that system (1.1) decouples to two system one normal to to other which can be written in geometric rather coordinate-free fashion, at least within the phases of solutions whereon the coefficients of the Aronsson system are continuous.

**Proposition 6.1.** *Let  $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . If  $K$  is the dilation (1.3) and its derivatives are given by (3.1) and (3.9), then the Aronsson system*

$$(6.1) \quad Q_\infty u = \left( K_P \otimes K_P + [K_P]^\perp K_{PP} \right) (Du) : D^2 u = 0$$

is equivalent on each phase  $\Omega_k = \text{int}\{rk(S(Du^\top Du)) = k\}$  to the pair of systems

$$(6.2) \quad S(\mathbf{G})D(\text{tr}(\mathbf{G})) = 0,$$

$$(6.3) \quad \mathbb{B}^\perp : (\text{tr}(\mathbf{G}))_P = 0,$$

where  $\mathbf{G}$  is given by (1.1),  $g = Du^\top Du$  is the Riemannian metric on  $u(\Omega)$ ,  $S$  is the Ahlfors operator and  $\mathbb{B}^\perp$  is the “generalized 2nd fundamental form”, defined for every local normal section  $\nu \in \Gamma([K_P(Du)]^\perp, D)$  over  $D \subseteq \Omega \setminus \mathcal{S}$  as  $(\mathbb{B}^\perp)_\nu := D\nu$ . Moreover, (6.2) is valid on all of  $\Omega$ .

We observe that system (6.2) can also be written as

$$(6.4) \quad S(g)D\left(\frac{\text{tr}(g)}{\det(g)^{1/n}}\right) = 0$$

and hence depends only on the metric structure of the immersion. System (6.2) is the “tangential system”. On the other hand, (6.3) can be written also as

$$(6.5) \quad \mathbb{B}^\perp : \left(\frac{\text{tr}(g)}{\det(g)^{1/n}}\right)_P = 0$$

and depends on the exterior geometry as well, the “shape” of  $u(\Omega)$ . System (6.3) is the “normal system”.

**Proof of Proposition 6.1.** By applying the orthogonal projections (4.2) and (4.3) to (6.1), we decouple it to

$$(6.6) \quad K_P(Du) \otimes K_P(Du) : D^2 u = 0,$$

$$(6.7) \quad [K_P(Du)]^\perp K_{PP}(Du) : D^2 u = 0.$$

In view of (3.1), we rewrite (6.6) as

$$(6.8) \quad Dug^{-1}S(g)D(K(Du)) = 0.$$

By using that  $K(Du) = \text{tr}(\mathbf{G})$  and that  $Dug^{-1}$  has constant rank equal to  $n$  and hence is left invertible, we obtain

$$(6.9) \quad (Dug^{-1})^{-1}Dug^{-1}S(g)D(\text{tr}(\mathbf{G})) = S(g)D(\text{tr}(\mathbf{G})) = 0.$$

Since  $g = \det(g)^{1/n}\mathbf{G}$ , system (6.9) leads to (6.2). To obtain (6.3), we observe that (6.7) is equivalent to

$$(6.10) \quad \nu^\top K_{PP}(Du) : D^2 u = 0,$$

for all local normal sections  $\nu \in \Gamma([K_P(Du)]^\perp, D)$ ,  $D \subseteq \Omega \setminus \mathcal{S}$ . By (5.22), equation (6.10) is equivalent to  $-D\nu : K_P(Du) = 0$ . Hence, we rewrite it as

$$(6.11) \quad -D\nu : (\text{tr}(\mathbf{G}))_P = 0.$$

By definition of  $\mathbb{B}^\perp$ , system (6.11) leads to (6.3) and the proposition follows.  $\square$

**Remark 6.2.** We will later show that the 2-dimensional case  $n = 2 \leq N$  is prominent. In this case, interfaces of discontinuities of the coefficients disappear and  $\mathbb{B}^\perp$  coincides with the standard 2nd fundamental form.

**Corollary 6.3** (Constant dilation on  $\Omega_n$ ). *Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$  solving  $K_P(Du) \otimes K_P(Du) : D^2u = 0$ . Then, on the  $n$ -phase  $\Omega_n$  given by (5.2),  $u$  has constant dilation on each connected component of  $\Omega_n$ .*

**Proof of Corollary 6.3.** By (5.2) and (6.9), we have that  $S(g)$  is invertible on  $\Omega_n$  and consequently we get  $D(K(Du)) = 0$  on  $\Omega_n$ .  $\square$

**6.2. Geometric Structure of Interfaces of Extremal  $\infty$ -Quasiconformal Maps.** We begin with a differential identity valid *on the interfaces* of discontinuity, under a local regularity assumption on the interface. We assume only  $C^1$  regularity, but we allow for possibly complicated topology and self-intersections.

**Proposition 6.4** (Covariant Derivatives on Interfaces). *Let  $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . Suppose the set of interfaces  $\mathcal{S}$  inside  $\Omega$  given by (5.3) contains a  $C^1$  immersed submanifold  $M$  and let  $\nabla^M$  be its Riemannian gradient. Then, we have the identity*

$$(6.12) \quad \begin{aligned} \nabla^M([K_P(Du)]^\perp) : K_P(Du) &= -([K_P]^\perp K_{PP})(Du) : D^2u \\ &+ ([K_P]^\perp K_{PP})(Du) : \nabla^{M^\perp} Du, \end{aligned}$$

valid on  $M \subseteq \mathcal{S}$ , where  $\nabla^{M^\perp}$  is the orthogonal complement of  $\nabla^M$  in  $\mathbb{R}^n$ .

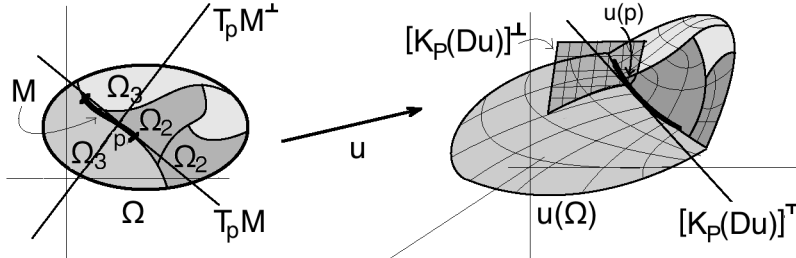


Figure 4.

**Remark 6.5.** The point in (6.12) is that  $[K_P(Du)]^\perp$  has covariantly differentiable contraction with  $K_P(Du)$  along (part of the interface)  $M$ , *without having assumed that  $S(Du^T Du)$  has constant rank on  $M$  and hence without having assumed that  $[K_P(Du)]^\perp$  is differentiable on  $M \subseteq \Omega$ .*

**Proof of Proposition 6.4.** By assuming as we can that  $M$  is immersed by the inclusion into  $\Omega$ , we fix a point  $p \in M \subseteq \Omega$  and consider coordinates near  $p$  adapted to the immersion. Let  $\{\nabla_1^M, \dots, \nabla_n^M\}$  denote the  $n$  components of  $\nabla^M$  with respect to the standard coordinates of  $\mathbb{R}^n$ . By differentiating covariantly near  $p$  the identity

$$(6.13) \quad [K_P(Du)]_{\alpha\beta}^\perp K_{P\beta j}(Du) = 0$$

we obtain

$$\begin{aligned} \nabla_i^M ([K_P(Du)]_{\alpha\beta}^\perp) K_{P\beta j}(Du) &= - [K_P(Du)]_{\alpha\beta}^\perp \nabla_i^M (K_{P\beta j}(Du)) \\ (6.14) \qquad \qquad \qquad &= - [K_P(Du)]_{\alpha\beta}^\perp K_{P\beta j P_{\gamma k}}(Du) \nabla_i^M D_k u_\gamma. \end{aligned}$$

By applying the expansion  $\nabla^M = D - \nabla^{M^\perp}$ , putting  $i = j$  and summing, (6.14) implies (6.12) and the proposition follows.  $\square$

The previous identity readily implies the next

**Corollary 6.6.** *In the setting of Proposition 6.4 above, if  $u$  solves the system  $([K_P]^\perp K_{PP})(Du) : D^2u = 0$ , then we have*

$$(6.15) \qquad \nabla^M ([K_P(Du)]^\perp) : K_P(Du) = ([K_P]^\perp K_{PP})(Du) : \nabla^{M^\perp} Du.$$

In particular, the vector field

$$(6.16) \qquad \nabla^M ([K_P(Du)]^\perp) : K_P(Du) : M \longrightarrow \mathbb{R}^N$$

is “normal” to  $u(M)$ , namely, it is valued in  $[K_P(Du)]^\perp$ :

$$(6.17) \qquad [K_P(Du)]^\top \left( \nabla^M ([K_P(Du)]^\perp) : K_P(Du) \right) = 0.$$

**Proof of Corollary 6.6.** Since the immersion  $u$  solves  $[K_P(Du)]^\perp K_{PP}(Du) : D^2u = 0$ , (6.12) gives (6.15). By applying the projection  $[K_P(Du)]^\top$  to the latter, (6.17) follows. Hence, the vector field  $\nabla^M ([K_P(Du)]^\perp) : K_P(Du)$  equals its projection on  $[K_P(Du)]^\perp$  and the corollary follows.  $\square$

## 7. SUFFICIENCY OF $K_P(Du) \otimes K_P(Du) : D^2u = 0$ FOR RANK-ONE LOCALLY MINIMAL DILATION WHEN $n = 2 \leq N$ .

In this section we show that in the case of 2-dimensional immersions when  $n = 2 \leq N$ , the tangential Aronsson system  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  is sufficient for the minimality notion of Rank-One Locally Minimal Dilation. This follows as a corollary of the fact that when  $n = 2$ , solutions of this system necessarily have constant dilation. In particular, the rank of  $S(Du^\top Du)$  is constant throughout the domain and interfaces of discontinuity on the coefficients of the normal system  $([K_P]^\perp K_{PP})(Du) : D^2u = 0$  disappear.

As a corollary, we show that when  $n = N = 2$ , the conjecture of Capogna-Raich in [CR] on the sufficiency of system  $(K_P \otimes K_P)(Du) : D^2u = 0$  for their stronger local minimality notion is false. This follows by Example 7.5 below in which we construct a diffeomorphism with constant dilation on a domain of the plane which has the same boundary values with the identity.

**Lemma 7.1** (Constant dilation). *Let  $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$  which solves  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  on  $\Omega$ . Suppose  $\Omega$  is connected and let  $\Omega_0^*, \dots, \Omega_n^*$  be the augmented  $n + 1$  phases of the immersion given by (5.4). Then:*

(i)  $S(Du^\top Du)$  has nowhere rank equal to one:

$$(7.1) \qquad \Omega_1^* = \emptyset.$$

(ii) If moreover  $n = 2$ , then  $\Omega_0^* \in \{\emptyset, \Omega\}$ . That is,  $\Omega_0^*$  is either empty or equals the whole  $\Omega$ . Hence,  $u$  has constant dilation everywhere on  $\Omega$ :

$$(7.2) \qquad K(Du) \equiv k \geq 2.$$

If it happens that  $\Omega_0^* \neq \emptyset$ , then  $k = 2$  and in this case  $u$  is conformal on  $\Omega$ .

**Proof of Lemma 7.1.** (i) On  $\Omega_1^*$  we have  $\text{rk}(S(Du^\top Du)) = 1$  and also  $S(Du^\top Du) = S(Du^\top Du)^\top$ . Since  $S(Du^\top Du)$  is a rank-one symmetric matrix, there exist  $\lambda : \Omega_1^* \rightarrow \mathbb{R}$  and  $a : \Omega_1^* \rightarrow \mathbb{R}^n$  such that  $\lambda > 0$ ,  $|a| = 1$  and  $S(Du^\top Du) = \lambda a \otimes a$ . Hence, we obtain

$$(7.3) \quad \lambda = \lambda |a|^2 = \text{tr}(\lambda a \otimes a) = \text{tr}(S(Du^\top Du)) = 0.$$

Consequently,  $\Omega_1^* = \emptyset$ .

(ii) When  $n = 2$ , by (i) we have that  $\Omega = \Omega_0^* \cup \Omega_2^*$ . On  $\Omega_0^*$  the immersion  $u$  is conformal. By Corollary 6.3, on  $\Omega_2^*$   $u$  has constant dilation. Hence,  $u$  has constant dilation on each connected component of  $\Omega_0^* \cup \Omega_2^* = \Omega$ . This means that  $K(Du)$  is piecewise constant on  $\Omega$ . By assumption,  $\Omega$  is connected and also  $K(Du) \in C^0(\Omega)$ . As a result, necessarily either  $\Omega_0^* = \emptyset$  or  $\Omega_0^* = \Omega$ . If  $\Omega_0^* \neq \emptyset$ , then  $u$  is conformal on  $\Omega$ . The lemma follows.  $\square$

**Proposition 7.2** (Equivalences in the 2-Dimensional case). *Let  $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . Then, the following are equivalent:*

- (i)  $u$  has Rank-One Locally Minimal Dilation on  $\Omega$ .
- (ii)  $u$  solves  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  on  $\Omega$ .
- (iii)  $u$  has constant dilation on connected components of  $\Omega$ .

**Proof of Proposition 7.2.** The implications (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii) have already been established, so it suffices to prove (iii)  $\Rightarrow$  (i). For, suppose  $u$  has constant dilation on connected components of  $\Omega$ . Fix  $D \subset \subset \Omega$ ,  $f \in C_0^1(D)$  and  $\xi \in \mathbb{S}^{N-1}$ . We may assume  $D$  is connected and that  $\text{rk}(Du + \xi \otimes Df) = n$  on  $D$ . Then, since  $f|_{\partial D} \equiv 0$ , there exists an interior critical point  $\bar{x} \in D$  of  $f$ . By using that  $Df(\bar{x}) = 0$ , we estimate

$$(7.4) \quad \begin{aligned} K_\infty(u + f\xi, D) &= \sup_D K(Du + \xi \otimes Df) \\ &\geq K(Du(\bar{x}) + \xi \otimes Df(\bar{x})) \\ &= K(Du(\bar{x})) \\ &= \sup_D K(Du) \\ &= K_\infty(u, D). \end{aligned}$$

Hence,  $u$  has rank-one locally minimal dilation and the proposition follows.  $\square$

Directly from Proposition 7.2 we obtain the following

**Corollary 7.3** (Absence of Interfaces in the 2-Dimensional case). *Let  $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$  which solves  $Q_\infty u = 0$  on the connected set  $\Omega$ . Then the rank of  $S(Du^\top Du)$  is constant on  $\Omega$ , and equals either 0 or 2. If  $\text{rk}(S(Du^\top Du)) = 0$  then  $u$  satisfies*

$$(7.5) \quad K(Du) \equiv 2,$$

$$(7.6) \quad K_{PP}(Du) : D^2u = 0.$$

The condition  $K(Du) \equiv 2$  is equivalent to Conformality:  $Du^\top Du = \frac{1}{n}|Du|^2 I$ . If  $\text{rk}(S(Du^\top Du)) = 2$ , then  $u$  satisfies

$$(7.7) \quad K(Du) \equiv \text{const.} > 2,$$

$$(7.8) \quad [Du]^\perp K_{PP}(Du) : D^2 u = 0.$$

**Remark 7.4.** Since the dilation (1.3) fails to be convex, it seems that sufficiency of the normal system  $[K_P(Du)]^\perp K_{PP}(Du) : D^2 u = 0$  for minimally distorted area does not hold. In particular, the respective convexity arguments used in the case of the  $\infty$ -Laplacian in [K3] fail.

The following example certifies that the variational notion of rank-one locally minimal dilation is genuinely weaker than the respective notion of “locally minimal dilation” used in [CR], where general vector-valued variations with the same boundary values are considered.

**Example 7.5** (Rank-One Locally Minimal Dilation is Strictly Weaker Notion). (cf. [CR], Cor 1.6(2)) Let  $\Omega := \mathbb{D}^2 \setminus \{0\} \subseteq \mathbb{R}^2$  be the punctured unit disc on the plane. Fix  $\gamma > -1$  and consider the maps  $u, u^\gamma : \Omega \rightarrow \Omega$  where  $u(x) := x$  and  $u^\gamma(x) := |x|^\gamma x$ . Then,  $u = u^\gamma$  on  $\partial\Omega = \mathbb{S}^1 \cup \{0\}$  and  $u$  is conformal on  $\Omega$  while  $u^\gamma$  is quasiconformal but has constant strictly greater dilation:

$$(7.9) \quad K(Du) \equiv 2 < 2 + \frac{\gamma^2}{\gamma + 1} \equiv K(Du^\gamma).$$

For completeness, we provide some details of our calculations. We readily have

$$(7.10) \quad Du^\gamma(x) = |x|^\gamma \left( I + \gamma \frac{x}{|x|} \otimes \frac{x}{|x|} \right)$$

and by setting  $\frac{x}{|x|} = (a, b)^\top$  we obtain

$$(7.11) \quad Du^\gamma(x) = |x|^\gamma \begin{bmatrix} 1 + \gamma a^2 & \gamma ab \\ \gamma ba & 1 + \gamma b^2 \end{bmatrix}.$$

By using that  $a^2 + b^2 = 1$ , we have

$$(7.12) \quad \begin{aligned} K(Du^\gamma) &= \frac{|Du^\gamma|^2}{(\det(Du^\gamma) \det(Du^\gamma))^{1/2}} \\ &= \frac{|x|^{2\gamma} [(1 + \gamma a^2)^2 + (1 + \gamma b^2)^2 + 2(\gamma ab)^2]}{|x|^{2\gamma} [(1 + \gamma a^2)(1 + \gamma b^2) - (\gamma ab)^2]} \\ &= 2 + \frac{\gamma^2}{\gamma + 1}. \end{aligned}$$

As a conclusion, in view of Corollary 7.2,  $u^\gamma$  has rank-one minimal dilation over  $\Omega$ , but does not have minimal dilation over  $\Omega$  since it has the same boundary values on  $\partial\Omega$  with a conformal map. If moreover  $\gamma > 0$ , then both  $u, u^\gamma$  are in  $C^1(\bar{\Omega})^2$ .

**7.1. On the sufficiency of  $K_P(Du) \otimes K_P(Du) : D^2 u = 0$  for rank-one locally minimal dilation in the case of dimensions  $3 \leq n \leq N$ .** In this subsection we loosely discuss the much more complicated case of dimensions  $n \geq 3$ . In this case results are less sharp since Lemma 7.1 generally fails when  $n > 2$ .

To begin with, let  $u : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^N$  be an immersion in  $C^2(\Omega)^N$ . Obviously, we have  $\text{rk}(Du) = 3 \leq N$ . By Lemma 3.1 and Proposition 6.1, we may rewrite

system  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  as

$$(7.13) \quad g^{-1}S(g)D(K(Du)) = 0,$$

where  $g = Du^\top Du$ . We recall that in the case of  $n = 2$ , Lemma 7.1 asserts that  $S(g)$  either has two nonzero opposite eigenvalues (and hence has a saddle structure), or it vanishes. In the two-dimensional case this covers all possible values of rank and it follows that the dilation is constant throughout connected domains.

When  $n = 3$ , Lemma 7.1 still works with the same proof, but now asserts only that

- (i) there is no one-dimensional phase  $\Omega_1^*$ , and
- (ii)  $\Omega = \Omega_0^* \cup \Omega_2^* \cup \Omega_3^*$  with  $K(Du)$  constant on connected components of the set  $\Omega_0^* \cup \Omega_3^*$ .

When  $n = 3$  no information is provided for the two-dimensional phase  $\Omega_2^*$ . Let us analyse more closely what happens in this case when  $\Omega_2^* \neq \emptyset$  and nontrivial interfaces of discontinuities may appear, where  $\Omega_2^* = \{\text{rk}(S(g)) = 2\}$ . Let  $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$  be the eigenvalue functions on  $\Omega$  of the Riemannian metric  $g$ . Then, the spectrum of  $S(g)$  is

$$(7.14) \quad \begin{aligned} \sigma(S(g)) &= \sigma(g) - \frac{\text{tr}(g)}{3} \\ &= \left\{ \lambda_1 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}, \lambda_2 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}, \lambda_3 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \right\} \\ &= \left\{ \frac{2\lambda_1 - \lambda_2 - \lambda_3}{3}, \frac{2\lambda_2 - \lambda_3 - \lambda_1}{3}, \frac{2\lambda_3 - \lambda_2 - \lambda_1}{3} \right\}. \end{aligned}$$

We distinguish the following cases:

- (a)  $0 < \lambda_1 = \lambda_2 = \lambda_3 =: \lambda$ . Then, by (7.14) we have that  $S(g) = 0$ .
- (b)  $0 < \lambda_1 = \lambda_2 =: \lambda < \lambda_3$ . Then, by (7.14) we have that

$$(7.15) \quad \sigma(S(g)) = \{-\mu, -\mu, 2\mu\}$$

where  $\mu := \frac{\lambda_3 - \lambda}{3} > 0$ . By the Spectral Theorem, there is an orthonormal frame  $\{a_1, a_2, a_3\}$  of  $\mathbb{R}^3$  such that

$$(7.16) \quad S(g) = -\mu(a_1 \otimes a_1 + a_2 \otimes a_2) + 2\mu a_3 \otimes a_3$$

and  $S(g)$  has rank three.

- (c)  $0 < \lambda_1 < \lambda_2 = \lambda_3$ . Again as before  $S(g)$  has rank three.

- (d)  $0 < \lambda_1 < \lambda_2 < \lambda_3$ . This is the only case where rank equal to two may appear.

Since  $\lambda_2 + \lambda_3 > 2\lambda_1$  and  $\lambda_1 + \lambda_2 < 2\lambda_3$ , we get

$$(7.17) \quad \mu_1 := \frac{2\lambda_1 - \lambda_2 - \lambda_3}{3} < 0, \quad \mu_3 := \frac{2\lambda_3 - \lambda_2 - \lambda_1}{3} > 0$$

but it may happen that

$$(7.18) \quad \mu_2 := \frac{2\lambda_2 - \lambda_3 - \lambda_1}{3}$$

vanishes, like for example in the extremal quasiconformal map  $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $u(x, y, z) := (e^x, \sqrt{2}ye^x, \sqrt{3}ze^x)^\top$ . We have

$$(7.19) \quad Du^\top Du(x, y, z) = e^{2x} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



and hence we get  $(\lambda_1, \lambda_2, \lambda_3) = (e^{2x}, 2e^{2x}, 3e^{2x})$ , which implies  $\mu_2 = 0$ . Generally, the set of interfaces of a three-dimensional extremal quasiconformal map is given by

$$(7.20) \quad \mathcal{S} = \partial\{\mu_2 = 0\}$$

and the two-dimensional phase of  $u$  is given by

$$(7.21) \quad \Omega_2 = \text{int}\{\mu_2 = 0\}.$$

Since  $S(g)$  is traceless, the condition  $\text{tr}(S(g)) = 0$  implies  $-\mu_1 = \mu_3 =: \mu > 0$  and hence  $\sigma(S(g)) = \{-\mu, 0, \mu\}$ . By the Spectral Theorem, there exists an orthonormal frame  $\{a, b, c\}$  of  $\mathbb{R}^3$  such that

$$(7.22) \quad S(g) = -\mu(a \otimes a - c \otimes c).$$

By (7.13), we have that  $D(K(Du))$  is perpendicular to  $\{a, c\}$  and hence

$$(7.23) \quad D(K(Du)) = b \otimes b D(K(Du))$$

which implies that the dilation of  $u$  varies only in the direction of  $b$ . Consequently,  $K(Du)$  depends only on  $b$  through a certain function  $k$ :

$$(7.24) \quad K(Du(x)) = k(b(x)).$$

Unlike the case  $n = 2$ , when  $n = 3$  we do *not* obtain that the dilation of three-dimensional extremal quasiconformal immersions is constant, at least not by the previous reasoning.

However, by Theorem 5.2 in all dimensions  $2 \leq n \leq N$  rank-one locally minimal dilation implies solvability of  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  and by the higher-dimensional extension of Example 7.5, rank-one locally minimal dilation is genuinely weaker than locally minimal dilation. Although it seems reasonable that  $K_P(Du) \otimes K_P(Du) : D^2u = 0$  is sufficient for rank-one locally minimal dilation, we can not definitely conclude for the validity of the conjecture of Capogna-Raich in [CR] for  $n \geq 3$ .

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